

HODGE INTEGRALS AND DEGENERATE CONTRIBUTIONS

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0. Introduction

0.1. Let X be a nonsingular, projective, 3 dimensional complex algebraic variety. Let $\overline{M}_{g_D, n}(X, \beta)$ be the moduli space of stable maps from genus g_D curves to X representing the homology class $\beta \in H_2(X, \mathbb{Z})$. The Gromov-Witten invariants of X are defined via tautological integrals over these moduli spaces of maps (against their virtual fundamental classes):

$$N_{\beta}^{g_D}(\gamma_1, \dots, \gamma_n) = \int_{[\overline{M}_{g_D, n}(X, \beta)]^{vir}} \prod_{i=1}^n \text{ev}_i^*(\gamma_i),$$

where ev_i is the i^{th} evaluation map and $\gamma_i \in H^*(X, \mathbb{Z})$. As the moduli spaces are Deligne-Mumford stacks, the Gromov-Witten invariants take values in \mathbb{Q} . Let T_X and K_X be the tangent bundle and the canonical class of X . For a 3-fold, the dimension formula shows the virtual dimensions do not depend upon the genus:

$$\dim_{vir}(\overline{M}_{g_D}(X, \beta)) = 3g_D - 3 + \chi(T_X) = -K_X \cdot \beta.$$

If we restrict attention to a fixed curve class $\beta \in H_2(X, \mathbb{Z})$, there are two basic possibilities: $-K_X \cdot \beta = 0$ or $-K_X \cdot \beta > 0$ (the negative case is of no interest here since then the Gromov-Witten invariants vanish). We will always take $\beta \neq 0$.

0.2. **Case $-K_X \cdot \beta = 0$.** If X is Calabi-Yau, this case holds for all classes β . Let d be a positive integer. Let $C \subset X$ be a nonsingular genus $g < g_D$ curve of class β/d . The moduli space $\overline{M}_{g_D}(X, \beta)$ contains a substack of maps with genus g_D domains which factor through a d -fold cover of C . Under suitable conditions, this substack of maps covering C is a connected component of $\overline{M}_{g_D}(X, \beta)$. In the latter case, the contribution of C to the genus g_D , class β Gromov-Witten invariant of X is well-defined. It is these degenerate contributions that are studied here. Degenerate contributions play a central role in identifying the integer quantities in the Gromov-Witten theory of X . These

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integrality properties remain a very mysterious part of the subject. In algebraic geometry, degenerate contributions are related to Hodge integrals over the moduli space of curves $\overline{M}_{g,n}$ [FP]. In string theory, recent progress in the study of these contributions has been made by a link to M-theory [GV1], [GV2] (see also [MM]). While the mathematical results presented here overlap with the M-theoretic results of [GV2], the precise connection between the two approaches is still not completely understood. The differences are discussed below in Section 0.3.

Let $C \subset X$ be a nonsingular genus g curve representing the class $[C] \in H_2(X, \mathbb{Z})$. For the degenerate analysis, we assume the normal bundle to C in X is general. Consider the moduli space of maps $\overline{M}_{g+h}(X, d[C])$. If $g = 0$ or 1 , this moduli space will have a connected component equal to $\overline{M}_{g+h}(C, d[C])$. The contribution $C_g(h, d)$ of C to the genus $g + h$ Gromov-Witten is thus well-defined for $g = 0, 1$ and all values $h \geq 0, d > 0$. The above component claim relies on rigidity arguments which possibly fail for multiple covers of genus $g \geq 2$ curves. However, in the degree 1 case, $\overline{M}_{g+h}(X, [C])$ has a component equal to $\overline{M}_{g+h}(C, [C])$ for all g and h . Hence, $C_g(h, 1)$ is always well-defined. At present, because of the possibility of deformations in X away from C , the correct definition of $C_g(h, d)$ in general is not known to the author.

The contributions in case $g = 0$ have recently been calculated in algebraic geometry [FP] and string theory [GV1], [MM]:

$$(1) \quad \sum_{h=0}^{\infty} C_0(h, 1) t^{2h} = \left(\frac{\sin(t/2)}{t/2} \right)^{-2},$$

$$(2) \quad C_0(h, d) = d^{2h-3} C_0(h, 1),$$

where $C \subset X$ is a nonsingular, rigid rational curve. The contribution

$$C_0(0, d) = 1/d^3$$

is the Aspinwall-Morrison formula which had been proven previously by several different methods [AM], [M], [V].

A nonsingular curve $C \subset X$ is *rigid* if $H^0(C, N) = 0$ where N is the normal bundle of C in X . For rational C , rigidity is equivalent to the bundle splitting $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Define $C \subset X$ to be *super-rigid* if, for all non-constant maps of nonsingular curves $\mu : C' \rightarrow C$,

$$H^0(C', \mu^*(N)) = 0.$$

It is clear rigidity and super-rigidity are equivalent in the rational case, but differ for higher genus. Super-rigidity is a generic condition on the normal bundle for elliptic curves in X . Kley has informed the author

his existence result for rigid elliptic curves on complete intersection Calabi-Yau 3-folds also shows the existence of super-rigid elliptic curves [K].

The contributions $C_1(0, d)$ are easily computed for super-rigid elliptic curves C . The component of the moduli space $\overline{M}_1(X, d[C])$ corresponding to maps with image C is nonsingular of dimension 0 (and equal to $\overline{M}_1(C, d[C])$). The points of $\overline{M}_1(C, d[C])$ correspond naturally to the set of subgroups of $\mathbb{Z} \oplus \mathbb{Z}$ of index d . Hence, after accounting for automorphisms,

$$C_1(0, d) = \frac{\sigma(d)}{d} = \sum_{i|d} \frac{1}{i}$$

(see, for example, [S]). All other contributions of an elliptic curve C vanish by the following result.

Theorem 1. *Let $C \subset X$ be a super-rigid elliptic curve. Then,*

$$C_1(h, d) = 0$$

for all $h > 0, d > 0$.

This vanishing was conjectured by Gopakumar-Vafa in [GV1] and is derived in M-theory in [GV2]. The proof given here uses basic constructions related to the virtual fundamental class.

The degree 1 contributions $C_g(h, 1)$ take a very simple form.

Theorem 2. *Let $g \geq 0$.*

$$\sum_{h=0}^{\infty} C_g(h, 1) t^{2h} = \left(\frac{\sin(t/2)}{t/2} \right)^{2g-2}.$$

Theorem 2 is derived in Section 2 by expressing the contributions $C_g(h, 1)$ as Hodge integrals over the moduli space of curves. The required integrals are then computed via geometric constructions, relations, and series manipulations. Theorem 2 is the main result of this paper.

The right side of Theorem 2 was encountered before in the following related result of [FP]:

$$(3) \quad 1 + \sum_{h \geq 1} \sum_{i=0}^h t^{2h} k^i \int_{\overline{M}_{h,1}} \psi_1^{2h-2+i} \lambda_{h-i} = \left(\frac{\sin(t/2)}{t/2} \right)^{-k-1}.$$

Theorem 2 gives an interpretation of these Hodge integrals in the Gromov-Witten theory of Calabi-Yau 3-folds.

0.3. M-theory predictions. The method of [GV1], [GV2] is to consider limits of type IIA string theory which may be conjecturally analyzed in M-theory. A remarkable proposal is made in [GV2] for the form of the Gromov-Witten potential \tilde{F} of a Calabi-Yau 3-fold X . Let

$$\tilde{F}(t, q) = \sum_{g \geq 0} t^{2g-2} \tilde{F}_g(t, q),$$

$$\tilde{F}_g(t, q) = \sum_{0 \neq \beta \in H_2(X, \mathbb{Z})} N_\beta^g q^\beta,$$

where N_β^g is the genus g Gromov-Witten invariant of X in curve class β . The potential \tilde{F} differs from the *full* potential by the constant map ($\beta = 0$) contribution – the constant contributions have been calculated in [FP], [GV1], [MM]. For each curve class $\beta \in H_2(X, \mathbb{Z})$ and genus g , there is an *integer* n_β^g counting BPS states in the associated M-theory. The formula of [GV2] is:

$$(4) \quad \tilde{F}(t, q) = \sum_{g, \beta} n_\beta^g t^{2g-2} \sum_{d > 0} \frac{1}{d} \left(\frac{\sin(dt/2)}{t/2} \right)^{2g-2} q^{d\beta}.$$

If $C_g^M(h, d)$ denotes the contribution of a single BPS state in genus g and class β to the Gromov-Witten invariant in genus $g + h$ and class $d\beta$, then formula (4) is equivalent to the equations:

$$\sum_{h=0}^{\infty} C_g^M(h, 1) t^{2h} = \left(\frac{\sin(t/2)}{t/2} \right)^{2g-2},$$

$$C_g^M(h, d) = d^{2g+2h-3} C_g^M(h, 1).$$

The first of these agrees with Theorem 2, so $C_g^M(h, 1) = C_g(h, 1)$. The second is a generalization of (2) to $g \geq 0$. It is therefore reasonable to interpret the states n_β^0 as counting embedded (virtual) curves of genus 0 and degree β (even for the Calabi-Yau quintic these numbers n_β^0 are at best virtual because of the existence of Vainsencher's nodal rational curves). However, when specialized to genus 1, the second equation yields $C_1^M(0, d) = 1/d$ in contrast to $C_1(0, d) = \sigma(d)/d$. The (virtual) count of embedded genus 1 curves should be derived from \tilde{F}_1 via the multiple cover corrections $C_0(1, d)$ and $C_1(0, d)$ (as previously pursued in [BCOV]). Gromov-Witten theory would predict the resulting number to be virtually enumerative, and thus integral (this heuristic argument for integrality is not a proof). The M-theoretic perspective predicts a *different* correction of \tilde{F}_1 to yield integers via formula (4). Klemm has checked the two genus 1 integrality predictions both hold in low degrees for several Calabi-Yau 3 folds [Kl]. These integrality constraints are not

trivially dependent. No proofs of any of these integrality constraints are known to the author.

To find higher genus evidence for the formula (4), a direct computation of the potential \tilde{F} in the local Calabi-Yau case (\mathbf{P}^2 with canonical bundle) for low genera and degrees has been pursued by Klemm and Zaslow [KIZ]. The Gromov-Witten invariants (in all genera) may be computed in this case by the virtual localization formula of [GP] and the holomorphic anomaly equation [BCOV]. The integrality predicted by (4) is a nontrivial constraint which is verified in all calculations.

At this point, it is not clear how to define or compute the general contributions $C_g(h, d)$. One may hope a complete understanding of $C_g(h, d)$ will lead to an integrality property of the Gromov-Witten potential of X distinct from (4).

0.4. Case $-K_X \cdot \beta > 0$. In this case, the moduli spaces $\overline{M}_{g_D}(X, \beta)$ have positive virtual dimensions. The Gromov-Witten invariants $N_\beta^{g_D}(\gamma)$ of X then depend upon a vector of cohomology classes

$$\gamma = (\gamma_1, \dots, \gamma_k), \quad \gamma_i \in H^*(X, \mathbb{Z}).$$

Let $Y_i \subset X$ be general topological cycles dual to the classes γ_i . If we wish to identify integers in this Gromov-Witten theory, degenerate contributions again play a role.

Let us assume we are in an ideal situation with respect to the moduli spaces of maps to X . Let $M_g^{Bir}(X, \beta)$ denote the moduli space of birational maps from smooth genus g domain curves. We assume first:

- (i) The spaces $M_g^{Bir}(X, \beta)$ are generically reduced and of the expected dimension for all $g \leq g_D$.

There is then an enumerative integer $E_\beta^{g_D}(\gamma)$ defined to equal the number of genus g_D maps of class β with smooth domains meeting all the cycles Y_i . However, $E_\beta^{g_D}(\gamma)$ will not equal $N_\beta^{g_D}(\gamma)$. The difference arises from the following observation. For each $g < g_D$, there are $E_\beta^g(\gamma)$ maps with smooth genus g domains of class β satisfying the required incidence conditions. The Gromov-Witten invariant $N_\beta^{g_D}(\gamma)$ receives a degenerate contribution from each of these lower genus solutions (via reducible genus g_D maps factoring through covers of the lower genus curves). As the genus g solution represents the class β , the covers must be of degree 1. These degenerate contributions are therefore analogous to $C_g(g_D - g, 1)$.

Dimension counts show maps multiple onto their image and maps with reducible images are not *expected* to contribute to $N_\beta^{g_D}(\gamma)$. This is the second ideal assumption:

- (ii) Maps in $\overline{M}_{g_D}(X, \beta)$ multiple onto their image or with reducible image do not satisfy incidence conditions to the cycles Y_i .

Let $C \subset X$ be a nonsingular, genus g curve of class β satisfying incidence conditions to the cycles Y_i . Assume further C is infinitesimally rigid with respect to these incidence conditions. The contribution $C_g(h, X, \beta)$ of C to the Gromov-Witten invariant $N_\beta^{g+h}(\gamma)$ is then well-defined: it is found in Section 3 to be an integral over the moduli space $\overline{M}_{g+h}(C, [C])$. This contribution is easily seen to be independent of γ . The final ideal assumption is:

- (iii) For all $g < g_D$, the solution maps counted by $E_\beta^g(\gamma)$ are nonsingular embeddings infinitesimally rigid with respect to the incidence conditions.

The ideal relation between Gromov-Witten theory and the enumerative invariants is:

$$(5) \quad N_\beta^{g_D}(\gamma) = \sum_{g=0}^{g_D} C_g(g_D - g, X, \beta) E_\beta^g(\gamma).$$

The validity of the relation (5) for $N_\beta^{g_D}(\gamma)$ requires assumptions (i), (ii), and (iii) for g_D , β , and γ .

The easiest 3-fold to consider is $X = \mathbf{P}^3$. As the divisor $-K_{\mathbf{P}^3}$ is ample, $-K_{\mathbf{P}^3} \cdot \beta > 0$ for all nonzero curve classes. The moduli spaces of maps to \mathbf{P}^3 are easily seen to be ideal in the above sense for the genera $g_D = 0, 1, 2$, all degrees $d > 0$, and all γ . The rigidity statements follow as usual from Bertini arguments (see [FuP]). Therefore, the ideal relation (5) holds in these genera. The equation

$$(6) \quad N_d^0 = E_d^0$$

is well known for \mathbf{P}^3 (we drop γ in these equations). In joint work with Getzler and Graber, we had computed

$$(7) \quad C_0(1, \mathbf{P}^3, d) = \frac{1-2d}{12},$$

$$N_d^1 = \frac{1-2d}{12} E_d^0 + E_d^1.$$

Equation (7) was used in Getzler's study [Ge] of the genus 1 enumerative geometry of \mathbf{P}^3 . Using Xiong's calculations of low degree genus 2 Gromov-Witten invariants of \mathbf{P}^3 as data, Jinzenji and Xiong conjectured the contribution equation:

$$(8) \quad N_d^2 = \frac{3-11d+10d^2}{720} E_d^0 - \frac{4d}{24} E_d^1 + E_d^2.$$

These equations led Jinzenji and Xiong to recently conjecture a general formula [J] analogous to Theorem 2:

$$(9) \quad \sum_{h=0}^{\infty} C_g(h, X, \beta) t^{2h} = \left(\frac{\sin(t/2)}{t/2} \right)^{2g-2-K_X \cdot \beta}.$$

The contribution $C_g(h, X, \beta)$ is calculated here by the method used in the proof of Theorem 2.

Theorem 3. *The degenerate contributions $C_g(h, X, \beta)$ are determined by formula (9).*

Theorem 3 and relation (5) prove formulas (6), (7), (8) for $g = 0, 1, 2$ and all degrees $d > 0$ in \mathbf{P}^3 . For higher genera, it is known the space of curves in \mathbf{P}^3 may be of excess dimension. For example, the moduli space $\overline{M}_3(\mathbf{P}^3, 4)$ has a 17 dimensional component, but is expected to be 16 dimensional. The definition of enumerative invariants is therefore not clear from a space curve point of view. However, the invariants $E_\beta^g(\gamma)$ may still be *defined* by Theorem 3 from the Gromov-Witten invariants and equation (5). Perhaps an integrality property holds for $E_\beta^g(\gamma)$ in some general context.

Algebraic 3-folds are special in Gromov-Witten theory since the (virtual) dimensions of the moduli spaces of stable maps do not depend upon the genus. A similar uniform treatment of degenerate contributions in higher dimensions will require new ideas. Graber has carried out a related degenerate analysis in the genus 0 Gromov-Witten theory of the Hilbert scheme of 2 points of \mathbf{P}^2 [Gr].

0.5. Moduli of curves. The Hodge integral approach taken here has an application to the geometry of the moduli space of nonsingular curves M_g , ($g \geq 2$). The tautological ring $\mathcal{R}^*(M_g)$ is the subring of $A^*(M_g)$ generated by the κ classes (see [Mu]). The intersection calculus of $\mathcal{R}(M_g)$ has a very rich structure. A detailed study by Faber of $\mathcal{R}(M_g)$ for low genera has led to very precise conjectures of this ring structure [F1]. In particular, Faber has conjectured $\mathcal{R}^*(M_g)$ is a Gorenstein ring with socle in degree $g - 2$. In [GeP], the (conjectural) intersection pairing of $\mathcal{R}(M_g)$ is directly linked to Gromov-Witten theory via (conjectural) Virasoro constraints on the descendent potential of \mathbf{P}^2 . The computation here of the degenerate contributions $C_g(h, 1)$ leads to a formula in $\mathcal{R}^*(M_g)$ conjectured previously by Faber from evidence for $g \leq 15$.

Theorem 4. *For $g \geq 2$, the relation*

$$\sum_{i=0}^{g-2} (-1)^i \lambda_i \kappa_{g-2-i} = \frac{2^{g-1}}{g!} \kappa_{g-2}$$

holds in $\mathcal{R}^(M_g)$.*

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1. Theorem 1

1.1. Super-rigidity. Let $C \subset X$ be a nonsingular elliptic curve in a Calabi-Yau 3-fold. The normal bundle N is of rank 2 with trivial determinant. If C is rigid, a straightforward argument shows N contains a non-trivial degree 0 line sub-bundle L :

$$0 \rightarrow L \rightarrow N \rightarrow L^{-1} \rightarrow 0.$$

Conversely, such a filtration implies the rigidity of C . The curve C is super-rigid if and only if L is not a torsion element of the Picard group of C . While super-rigidity is a stronger condition on N than rigidity, it is an open condition. Super-rigidity is required for the equality of moduli spaces proven in Proposition 1. Note super-rigidity implies $H^0(C', \mu^*(N)) = 0$ for every non-constant *stable* map $\mu : C' \rightarrow C$.

The moduli spaces $\overline{M}_{1+h}(C, d[C])$ and $\overline{M}_{1+h}(X, d[C])$ are Deligne-Mumford stacks with possibly nonreduced structures.

Proposition 1. *Let $C \subset X$ be a nonsingular, super-rigid elliptic curve. The space of maps $\overline{M}_{1+h}(C, d[C])$ is a union of connected components of $\overline{M}_{1+h}(X, d[C])$ for all $h \geq 0$, $d > 0$.*

Proof. There is a natural map:

$$\iota : \overline{M}_{1+h}(C, d[C]) \rightarrow \overline{M}_{1+h}(X, d[C]).$$

By the super-rigidity of C , the locus of $\overline{M}_{1+h}(X, d[C])$ corresponding to maps with support in C is a union of connected components of $\overline{M}_{1+h}(X, d[C])$. We will prove ι is an isomorphism onto these connected components.

It suffices to prove a lifting property for families of stable maps over Artinian local rings A . Let $\xi \in \text{Spec}(A)$ be the geometric point corresponding to the maximal ideal $m \subset A$. Let

$$\pi : \mathcal{F} \rightarrow \text{Spec}(A), \quad \mu : \mathcal{F} \rightarrow X$$

be a family of stable maps satisfying

$$(10) \quad \mu_\xi : \mathcal{F}_\xi \rightarrow C \subset X.$$

We will prove μ factors through C . This lifting implies the desired isomorphism property of ι .

Let \mathcal{I} be the ideal sheaf of C in X . We must prove the natural map

$$\phi : \mu^*(\mathcal{I}) \rightarrow \mathcal{O}_{\mathcal{F}}$$

is zero. Certainly ϕ has image in $m\mathcal{O}_{\mathcal{F}}$ by the assumption (10) on the geometric fiber ξ . Hence, ϕ induces a natural map on \mathcal{F} :

$$(11) \quad \mu^*(\mathcal{I}/\mathcal{I}^2) \rightarrow m\mathcal{O}_{\mathcal{F}}/m^2\mathcal{O}_{\mathcal{F}} = (m/m^2) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{F}_\xi}.$$

The restriction of $\mu^*(\mathcal{I}/\mathcal{I}^2)$ to \mathcal{F}_ξ is simply $\mu_\xi^*(N^*)$. By the super-rigidity of C , the map (11) is zero. We conclude ϕ factors through $m^2\mathcal{O}_{\mathcal{F}}$.

The above argument may be used to prove the following implication: if ϕ factors through $m^k\mathcal{O}_{\mathcal{F}}$, then ϕ factors through $m^{k+1}\mathcal{O}_{\mathcal{F}}$. Since A is Artinian, m is nilpotent. Hence, ϕ vanishes. \square

There are two perfect obstruction theories on $\overline{M}_{1+h}(C, d[C])$ obtained from considering the moduli problem of maps to C and X respectively (see [B], [BF], [LT]). Let

$$\pi : \mathcal{F} \rightarrow \overline{M}_{1+h}(C, d[C]),$$

$$\mu : \mathcal{F} \rightarrow C$$

be the universal family and universal map respectively. By super-rigidity $\pi_*\mu^*(N) = 0$ and $R^1\pi_*\mu^*(N)$ is a rank $2h$ bundle. The two obstruction theories differ exactly by the bundle $R^1\pi_*\mu^*(N)$. From the definition of the virtual class, we conclude:

$$(12) \quad C_1(h, d) = \int_{[\overline{M}_{1+h}(C, d[C])]^{vir}} c_{2h}(R^1\pi_*\mu^*(N)).$$

1.2. Vanishing results. Let E be any bundle on C . Consider the complex $R\pi_*\mu^*(E)$ in the derived category of coherent sheaves on $\overline{M}_{1+h}(C, d[C])$. Let \mathcal{L} be a π -relatively ample polarization on \mathcal{F} . We may find an exact sequence of bundles on \mathcal{F} :

$$0 \rightarrow K \rightarrow \oplus \mathcal{L}^{-k} \rightarrow \mu^*(E) \rightarrow 0$$

for some positive integer k [H]. As both $\pi_* K$ and $\pi_* \mathcal{L}^{-k}$ vanish, we find a two term bundle resolution of $R\pi_* \mu^*(E)$:

$$[R^1 \pi_* K \rightarrow R^1 \pi_* \oplus \mathcal{L}^{-k}] \simeq R\pi_* \mu^*(E).$$

The Chern classes of $R\pi_* \mu^*(E)$ are defined by $c(R^1 \pi_* K)/c(R^1 \pi_* \oplus \mathcal{L}^{-k})$. This definition is independent of two term resolutions in the derived category.

As $\pi_* \mu^*(N) = 0$ and $R^1 \pi_* \mu^*(N)$ is a rank $2h$ bundle, we see (12) may now be rewritten as:

$$C_1(h, d) = \int_{[\overline{M}_{1+h}(C, d[C])]^{vir}} [c^{-1}(R\pi_* \mu^*(N))]_{2h}.$$

It is easy to find flat families of bundles on C connecting N and the the trivial rank 2 bundle $I = \mathcal{O}_C \oplus \mathcal{O}_C$. For example, if P is a sufficiently ample line bundle, both $N \otimes P$ and $I \otimes P$ will have nowhere vanishing sections:

$$0 \rightarrow \mathcal{O}_C \rightarrow N \otimes P \rightarrow P^2 \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}_C \rightarrow I \otimes P \rightarrow P^2 \rightarrow 0.$$

Hence N and I are connected in the family of extensions of P by P^{-1} . The integral

$$\int_{[\overline{M}_{1+h}(C, d[C])]^{vir}} [c^{-1}(R\pi_* \mu^*(E))]_{2h}$$

is clearly constant as E varies in this family (for example, the two term resolutions of $R\pi_* \mu^*(E)$ may be chosen compatibly over the family). We conclude,

$$C_1(h, d) = \int_{[\overline{M}_{1+h}(C, d[C])]^{vir}} [c^{-1}(R\pi_* \mu^*(I))]_{2h}$$

Now assume $h > 0$. Let $\gamma : \overline{M}_{1+h}(C, d[C]) \rightarrow \overline{M}_{1+h}$ be the natural map to the moduli space of curves. Let \mathbb{E} denote the Hodge bundle on \overline{M}_{1+h} : the fiber of \mathbb{E} over the moduli point $[F] \in \overline{M}_{1+h}$ is $H^0(F, \omega_F)$ (see [Mu]). Since

$$\pi_* \mu^*(I) = \mathcal{O}_{\overline{M}} \oplus \mathcal{O}_{\overline{M}},$$

$$R^1 \pi_* \mu^*(I) = \gamma^*(\mathbb{E}^* \oplus \mathbb{E}^*),$$

we see $[c^{-1}(R\pi_* \mu^*(I))]_{2h}$ is a cohomology class pulled-back via γ from \overline{M}_{1+h} . Hence, to complete the proof of Theorem 1, it suffices to show the following vanishing.

Proposition 2. *Let $h > 0$. Then, $\gamma_*([\overline{M}_{1+h}(C, d[C])]^{vir}) = 0$.*

Proof. Fix a base point $p \in C$ for the course of the proof. We will consider the moduli space of 1-pointed maps $\overline{M}_{1+h,1}(C, d[C])$. Let

$$\text{ev}_1^{-1}(p) = \overline{M}_{1+h,p}(C, d[C]) \subset \overline{M}_{1+h,1}(C, d[C])$$

denote the subspace of maps for which the marking has image p . There is a canonical isomorphism obtained by the group law on C :

$$\overline{M}_{1+h,1}(C, d[C]) \cong C \times \overline{M}_{1+h,p}(C, d[C]).$$

Let $\rho : \overline{M}_{1+h,1}(C, d[C]) \rightarrow \overline{M}_{1+h,p}(C, d[C])$ denote the canonical projection.

The perfect obstruction theory on $\overline{M}_{1+h,1}(C, d[C])$ may be obtained from a canonical distinguished triangle involving the cotangent complex of the Artin stack of prestable curves and the perfect obstruction theory relative to this Artin stack (see [B], [BF], [GrP]). These objects are naturally equivariant with respect to the natural group law on C (see the constructions of [B], [BF]). Hence, the virtual class of $\overline{M}_{1+h,1}(C, d[C])$ is a pull-back of an algebraic cycle class on $\overline{M}_{1+h,p}(C, d[C])$. As the map

$$\gamma_1 : \overline{M}_{1+h,1}(C, d[C]) \rightarrow \overline{M}_{1+h,1}$$

factors through $\overline{M}_{1+h,p}(C, d[C])$, we obtain

$$(13) \quad \gamma_{1*}([\overline{M}_{1+h,1}(C, d[C])]^{vir}) = 0.$$

Consider now the commutative diagram obtained from the 1-pointed moduli spaces:

$$(14) \quad \begin{array}{ccc} \overline{M}_{1+h,1}(C, d[C]) & \xrightarrow{\gamma_1} & \overline{M}_{1+h,1} \\ \pi \downarrow & & \pi \downarrow \\ \overline{M}_{1+h}(C, d[C]) & \xrightarrow{\gamma} & \overline{M}_{1+h}. \end{array}$$

While (14) is not a fiber square, it is easy to see the following equality holds

$$(15) \quad \gamma_{1*}\pi^* = \pi^*\gamma_*.$$

From the Axiom of contracting a point [BM], we see

$$\pi^*([\overline{M}_{1+h}(C, d[C])]^{vir}) = [\overline{M}_{1+h,1}(C, d[C])]^{vir}.$$

Then, equations (13) and (15) imply:

$$(16) \quad \pi^*\gamma_*([\overline{M}_{1+h}(C, d[C])]^{vir}) = 0.$$

For any class $\xi \in A_*(\overline{M}_{1+h})$,

$$\pi_*(\psi_1 \cdot \pi^*(\xi)) = 2h \cdot \xi,$$

where ψ_1 is the Chern class of the cotangent line on $\overline{M}_{1+h,1}$. Hence, the pull-back $\pi^* : A_*(\overline{M}_{1+h}) \rightarrow A_*(\overline{M}_{1+h,1})$ is injective. The Proposition now follows from (16). \square

2. Theorem 2

2.1. Rigidity. Let $C \subset X$ be a rigid, nonsingular genus g curve with normal bundle N . The contribution $C_g(0, 1)$ is certainly 1, so we may assume h is a positive integer. The proof of Proposition 1 also establishes the following facts. First, the moduli space $\overline{M}_{g+h}(C, [C])$ is a component (easily seen to be connected) of $\overline{M}_{g+h}(X, [C])$. Second, the contribution $C_g(h, 1)$ is determined by:

$$(17) \quad C_g(h, 1) = \int_{[\overline{M}_{g+h}(C, [C])]^{vir}} c_{2h}(R_{g,h}).$$

Here, $R_{g,h}$ denotes the rank $2h$ bundle $R^1\pi_*\mu^*(N)$. Note the virtual dimension of $\overline{M}_{g+h}(C, [C])$ is also $2h$. The arguments of Section 1 are valid because a rigid curve is super-rigid in degree 1.

2.2. Irreducible components of $\overline{M}_{g+h}(C, [C])$. Let C be a nonsingular genus g curve. Let h be a positive integer. We first analyze the moduli space of degree 1 maps $\overline{M}_{g+h}(C, [C])$. Let $P(h)$ denote the set of partitions h . There is a natural set-theoretic function:

$$\nu : \overline{M}_{g+h}(C, [C]) \rightarrow P(h)$$

defined by the following method. Let $\mu : F \rightarrow C$ correspond to a point $[\mu] \in \overline{M}_{g+h}(C, [C])$. The domain F must contain a unique irreducible component F_C mapped isomorphically to C by μ . The arithmetic genera of the connected components $\{F_i\}$ of $F \setminus F_C$ form a partition of h . Let $\nu([\mu])$ equal this partition. The irreducible components of $\overline{M}_{g+h}(C, [C])$ are in bijective correspondence with $P(h)$ by the value of ν on a general element.

Let $\tau = (h_1 \geq \dots \geq h_l)$ be a partition of h of length l . Consider the Fulton-MacPherson configuration space $C[l]$ of l marked points in C : $C[l]$ is a natural compactification of the space of l distinct points of C [FuM]. If C has no automorphisms, $C[l]$ is simply the fiber of $\overline{M}_{g,l} \rightarrow \overline{M}_g$ over the moduli point $[C]$. Define the nonsingular Deligne-Mumford stack I_τ by:

$$(18) \quad I_\tau = C[l] \times \prod_{i=1}^l \overline{M}_{h_i,1}.$$

There is a natural family,

$$\pi : \mathcal{F} \rightarrow I_\tau,$$

of prestable curves over I_τ obtained by attaching a 1-pointed genus h_i curve to the i^{th} marking of the universal l -pointed curve over $C[l]$. Moreover, there is canonical projection $\mu : \mathcal{F} \rightarrow C$. The induced morphism:

$$\gamma_\tau : I_\tau \rightarrow \overline{M}_\tau \subset \overline{M}_{g+h}(C, [C])$$

is finite and surjective onto the irreducible component \overline{M}_τ corresponding to the partition τ .

Let $\partial \overline{M}_{h_i,1}$ denote the boundary of the moduli space: the locus of curves with at least one node. Similarly, let $\partial C[l] \subset C[l]$ denote the locus of nodal curves ($\partial C[l]$ may also be viewed as the locus lying over the diagonals of the product C^l). Let ∂I_τ denote the union of the pull-backs of the boundaries of the factors (18) via the $l+1$ projections. Let

$$\partial \gamma_\tau : \partial I_\tau \rightarrow \overline{M}_{g+h}(C, [C])$$

denote the natural map. The main geometric result used in the proof of Theorem 2 is the following vanishing.

Proposition 3. *For all partitions τ of h ,*

$$c_{2h}(\partial \gamma_\tau^*(R_{g,h})) = 0.$$

Proof. By definition, ∂I_τ is union of the pull-backs of the boundary divisors of the $l+1$ product factors of (18). We show $c_{2h}(\partial \gamma_\tau^*(R_{g,h}))$ restricts to 0 on each of these pull-backs.

Let pr_j denote the projection of I_τ onto the $(j+1)^{st}$ factor of (18) for $0 \leq j \leq l$. There are l natural evaluation maps $\text{ev}_i : C[l] \rightarrow C$ obtained from the l markings. Define $\text{ev}_i : I_\tau \rightarrow C$ by the composition

$$I_\tau \xrightarrow{\text{pr}_0} C[l] \xrightarrow{\text{ev}_i} C$$

for $1 \leq i \leq l$.

The bundle $\gamma_\tau^*(R_{g,h})$ is easily analysed via the natural normalization sequence of the family \mathcal{F} . We find a decomposition:

$$(19) \quad \gamma_\tau^*(R_{g,n}) = \bigoplus_{i=1}^l \mathbb{E}_i^* \otimes \text{ev}_i^*(N)$$

where \mathbb{E}_i is the Hodge bundle on $\overline{M}_{h_i,1}$. We denote the pull-back of these Hodge bundles to I_τ by the same symbols. An important relation among the Chern classes of the Hodge bundle has been established by Mumford in [Mu]. Mumford's relation is: $c(\mathbb{E}_i) \cdot c(\mathbb{E}_i^*) = 1$ in $A^*(\overline{M}_{h_i,1})$.

From (19), we deduce:

$$c_{2h}(\gamma_\tau^*(R_{g,h})) = \prod_{i=1}^l c_{2h_i}(\mathbb{E}_i^* \otimes \text{ev}_i^*(N)).$$

Algebra and Mumford's relation then yield:

$$(20) \quad c_{2h}(\gamma_\tau^*(R_{g,h})) = \prod_{i=1}^l \lambda_{h_i} \lambda_{h_i-1} c_1(\text{ev}_i^*(N^*)).$$

Here, λ_k denotes the k^{th} Chern class of the Hodge bundle.

First, consider a boundary divisor $\Delta \subset \overline{M}_{h_j,1}$. The pull-back of Δ to I_τ is simply:

$$\text{pr}_j^{-1}(\Delta) = C[l] \times \Delta \times \prod_{i \neq j} \overline{M}_{h_i,1}.$$

The restriction of the factor $\lambda_{h_j} \lambda_{h_j-1}$ of (20) to Δ has been proven by Faber to vanish [F1] (the reducible divisors of $\overline{M}_{h_1,1}$ have non-trivial genus splittings). Hence, the restriction of $c_{2h}(\gamma_\tau^*(R_{g,h}))$ to $\text{pr}_j^{-1}(\Delta)$ vanishes.

Second, consider a boundary divisor Δ of $C[l]$. The divisor Δ corresponds to a locus in which a subset $S \subset [l]$ (of at least 2 elements) of the marked points coincide over C . The evaluation maps $\{\text{ev}_i\}_{i \in S}$ coincide when restricted to $\text{pr}_0^{-1}(\Delta)$. Therefore, since $c_1(N^*)^2 = 0$, the restriction of $c_{2h}(\gamma_\tau^*(R_{g,h}))$ to $\text{pr}_j^{-1}(\Delta)$ vanishes. \square

2.3. Hodge integrals. Let $\partial \overline{M}_\tau = \gamma_\tau(\partial I_\tau)$, and let $M_\tau = \overline{M}_\tau \setminus \partial \overline{M}_\tau$. M_τ is open in $\overline{M}_{g+h}(C, [C])$ and corresponds to the moduli space of degree 1 maps which consist of nonsingular curves of genus h_i attached to distinct point of C .

A deformation theory argument shows M_τ is a nonsingular moduli stack of dimension $\sum_{i=1}^l (3h_i - 1)$. More precisely, for $[\mu : F \rightarrow C] \in M_\tau$, there is a canonical exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Aut}_{[F]} \rightarrow H^0(F, \mu^*(T_C)) \rightarrow \text{Def}_{[\mu]} \xrightarrow{\iota} \\ \text{Def}_{[F]} \rightarrow H^1(F, \mu^*(T_C)) \rightarrow \text{Obs}_{[\mu]} \rightarrow 0 \end{aligned}$$

where $\text{Aut}_{[F]}$ is the infinitesimal automorphisms of F and $\text{Def}_{[F]}$, $\text{Def}_{[\mu]}$ are the infinitesimal deformation spaces of F , μ respectively. It is easy to prove the cokernel of ι is equal to a vector space V with filtration

$$0 \rightarrow \text{Def}_{[C]} \rightarrow V \rightarrow \bigoplus_{i=1}^l (T_{p_i} \otimes T_{p'_i}) \rightarrow 0.$$

Here, the component $F_i \subset F$ of genus h_i is attached to C at the points $p_i \in F_i$ and $p'_i \in C$. The cokernel computation amounts to showing the map μ has no infinitesimal deformations which smooth any of the l nodes of F . We then see $\text{Def}_{[\mu]}$ is of constant dimension $\sum_{i=1}^l (3h_i - 1)$. Moreover, the obstruction space is a bundle over M_τ with fiber

$$(21) \quad \frac{H^1(F, \mu^*(T_C))}{\text{Im}(V)} = \bigoplus_{i=1}^l \frac{H^1(F_i, \mathcal{O}_{F_i} \otimes T_{p'_i})}{T_{p_i} \otimes T_{p'_i}}$$

over $[\mu]$. The essential point here is the deformation theory of maps in M_τ is very simple.

Let Aut_τ denote the stabilizer of the permutation \mathbb{S}_l -action on the l -tuple τ . The map $\gamma_\tau : I_\tau \rightarrow \overline{M}_\tau$ is Aut_τ -invariant. Moreover, the quotient map induces a proper, bijective morphism

$$(22) \quad \tilde{\gamma}_\tau : I_\tau / \text{Aut}_\tau \rightarrow \overline{M}_\tau.$$

Let $I_\tau^0 = I_\tau \setminus \partial I_\tau$. Certainly, $\tilde{\gamma}_\tau$ induces an isomorphism $I_\tau^0 / \text{Aut}_\tau \cong M_\tau$.

The restriction of the virtual class $\xi^{vir} = [\overline{M}_{g+h}(C, [C])]^{vir}$ to the disjoint open union $\bigcup_{\tau \in P(h)} M_\tau$ is:

$$\bigoplus_{\tau \in P(h)} \xi_\tau^{vir},$$

where $\xi_\tau^{vir} \in A_{2h}(M_\tau)$. The pull-back of ξ_τ^{vir} to I_τ^0 is identified from the obstruction theory (21) to be:

$$(23) \quad \gamma_\tau^*(\xi_\tau^{vir}) = \prod_{i=1}^l c_{h_i-1} \left(\frac{c(\mathbb{E}_i^* \otimes \text{ev}_i^*(T_C))}{1 - \psi_1 + c_1(\text{ev}_i^*(T_C))} \right).$$

Since M_τ is nonsingular, the restriction of the virtual class is the Euler class of the obstruction bundle.

The virtual class ξ^{vir} may be (non-canonically) expressed as a sum:

$$\bigoplus_{\tau \in P(h)} \overline{\xi}_\tau^{vir},$$

where $\overline{\xi}_\tau^{vir} \in A_{2h}(\overline{M}_\tau)$. Using the proper bijection (22), we see:

$$(24) \quad C_g(h, 1) = \sum_{\tau \in P(h)} \int_{I_\tau / \text{Aut}_\tau} \overline{\xi}_\tau^{vir} \cap c_{2h}(\tilde{\gamma}_\tau^*(R_{g,h})).$$

By the vanishing of Proposition 3, equation (24) remains valid if $\overline{\xi}_\tau^{vir}$ is replaced with *any* cycle class which restricts to ξ_τ^{vir} on M_τ . This

observation together with (23) yields the equality:

(25)

$$C_g(h, 1) = \sum_{\tau \in P(h)} \frac{1}{|\text{Aut}_\tau|} \int_{I_\tau} c_{2h}(\gamma_\tau^*(R_{g,h})) \cdot \prod_{i=1}^l c_{h_i-1} \left(\frac{c(\mathbb{E}_i^* \otimes \text{ev}_i^*(T_C))}{1 - \psi_1 + c_1(\text{ev}_i^*(T_C))} \right).$$

Equation (20) together with basic algebraic manipulations then prove the main integral formula:

(26)

$$C_g(h, 1) = \sum_{\tau \in P(h)} \frac{(2-2g)^l}{|\text{Aut}_\tau|} \prod_{i=1}^l \int_{M_{h_i,1}} \lambda_{h_i} \lambda_{h_i-1} \left(\sum_{j=0}^{h_i-1} (-1)^j \lambda_j \psi_1^{h_i-1-j} \right).$$

The only aspect of N which affects the integral

$$C_g(h, 1) = \int_{[\overline{M}_{g+h}(C, [C])]^{vir}} c_{2h}(R_{g,h}).$$

is $\int_C c_1(N^*)$. This Chern class enters enters (26) via equation (20) yielding the factor

$$\left(\int_C c_1(N^*) \right)^l = (2-2g)^l.$$

Theorems 2-4 will directly follow from formula (26).

For $q \geq 1$, define $\alpha_q = \int_{\overline{M}_{q,1}} \lambda_q \lambda_{q-1} \left(\sum_{j=0}^{q-1} (-1)^j \lambda_j \psi_1^{q-1-j} \right)$. Define the generating series:

$$Q(t) = \sum_{q \geq 1} \alpha_q t^{2q}.$$

An immediate consequence of formula (26) is the equation:

$$\begin{aligned} \sum_{h \geq 0} C_g(h, 1) t^{2h} &= \exp((2-2g)Q(t)) \\ &= \exp(2Q(t))^{1-g} \\ &= \left(\sum_{h \geq 0} C_0(h, 1) t^{2h} \right)^{1-g} \\ &= \left(\frac{\sin(t/2)}{t/2} \right)^{2g-2}. \end{aligned}$$

The last equality follows from the previous computations of $C_0(h, 1)$ in [FP]. The proof of Theorem 2 is complete.

3. Theorem 3

We follow here the notation of Section 0.4 . Let C be a nonsingular genus g curve in a 3-fold X representing the homology class β . We now assume $-K_X \cdot \beta > 0$, so the moduli space $\overline{M}_g(X, \beta)$ is of positive expected dimension. Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a vector of cohomology classes defining a Gromov-Witten invariant $N_\beta^g(\gamma)$. For each i , Let $Y_i \subset X$ be a topological cycle dual to γ_i . Let $p_i \in C \cap Y_i$. We let (C) denote the identity map $\pi : C \rightarrow C \subset X$ defining a point in the moduli space of stable maps. The contribution of C to $N_\beta^{g+h}(\gamma)$ via covers will require two general position hypotheses analogous to rigidity in the Calabi-Yau case:

- (i) (C, p_1, \dots, p_n) is a nonsingular point of $\overline{M}_{g,n}(X, \beta)$ lying on a component of expected dimension $-K_X \cdot \beta + n$.
- (ii) The topological intersection of the cycles $\text{ev}_i^{-1}(Y_i)$ in $\overline{M}_{g,n}(X, \beta)$ is transverse at (C, p_1, \dots, p_n) .

Under these hypotheses, the degenerate contribution of C may be expressed directly as an integral over $\overline{M}_{g+h}(C, [C])$.

Let $W \subset \overline{M}_g(X, \beta)$ be the open, nonsingular, expected dimensional subset of the moduli space of maps. Let $U \subset W$ be the open subset corresponding to embeddings of nonsingular genus g curves in X . As such embeddings have no nontrivial automorphisms, U is a nonsingular variety (not just a Deligne-Mumford stack). Moreover, by assumption (i), U is nonempty of dimension $-K_X \cdot \beta$ and contains (C) . After discarding a finite number of points of U , we may assume (C) is the only point of U meeting all the cycles Y_i . Note the moduli space U is also naturally an open set of a component of the Hilbert scheme of curves in X . Let

$$\eta : \mathcal{C} \rightarrow U$$

denote the universal family of curves over U . Let $\overline{M}_{g+h}(\eta, \beta)$ denote the η -relative moduli space of maps representing the fundamental class of the fibers of η . There is a natural morphism of Deligne-Mumford stacks:

$$(27) \quad \iota : \overline{M}_{g+h}(\eta, \beta) \rightarrow \overline{M}_{g+h}(X, \beta)$$

obtained by composition. There are several tautological morphisms (over U):

$$\pi : \mathcal{F} \rightarrow \overline{M}_{g+h}(\eta, \beta),$$

$$\mu : \mathcal{F} \rightarrow \mathcal{C},$$

$$\tau : \overline{M}_{g+h}(\eta, \beta) \rightarrow U.$$

Let \mathcal{N} denote the universal normal bundle \mathcal{N} on \mathcal{C} . \mathcal{N} is the family of normal bundles of the fibers of η in X . As U is nonsingular of expected dimension, $\eta_*(\mathcal{N})$ is isomorphic to the tangent bundle of U and $R^1\eta_*(\mathcal{N}) = 0$.

A deformation theoretic check over Artinian rings shows ι is an open immersion. We see the stack $\overline{M}_{g+h}(\eta, \beta)$ has two natural fundamental classes. The first is $[\overline{M}_{g+h}(\eta, \beta)]^{vir}$ obtained from the structure of a η -relative moduli space of maps. Second, the open inclusion ι endows $\overline{M}_{g+h}(\eta, \beta)$ with the perfect obstruction theory on $\overline{M}_{g+h}(X, \beta)$. A direct comparison of these two obstruction theories on $\overline{M}_{g+h}(\eta, \beta)$ shows they differ exactly by the bundle $R_{g,h} = R^1\pi_*\mu^*(\mathcal{N})$:

$$(28) \quad \iota^*([\overline{M}_{g+h}(X, \beta)]^{vir}) = [\overline{M}_{g+h}(\eta, \beta)]^{vir} \cap c_{2h}(R_{g,h}).$$

Relations (27) and (28) are valid when considered in the context of n -pointed stable maps (this may be deduced from the above unpointed relations together with the natural properties of these virtual structures under the morphisms forgetting the markings [BM]).

By relation (28) and the definition of the Gromov-Witten invariants, the contribution of (C, p_1, \dots, p_n) to $N_\beta^{g+h}(\gamma)$ is equal to the intersection product:

$$(29) \quad [\overline{M}_{g+h,n}(\eta, \beta)]^{vir} \cap c_{2h}(R_{g,h}) \cap \prod_{i=1}^n \text{ev}_i^{-1}(Y_i),$$

with value in the zeroth homology of the compact space $\cap_{i=1}^n \text{ev}_i^{-1}(Y_i)$. By assumption (ii) and the pull-back properties of the virtual class, intersection (29) is (numerically) equal to:

$$(30) \quad [\overline{M}_{g+h}(\eta, \beta)]^{vir} \cap c_{2h}(R_{g,h}) \cap \tau^{-1}(C).$$

The latter class (30) is an integral over the virtual class of the fiber $\tau^{-1}(C) = \overline{M}_{g+h}(C, [C])$. We find:

$$(31) \quad C_g(h, X, \beta) = \int_{[\overline{M}_{g+h}(C, [C])]^{vir}} c_{2h}(R_{g,h}).$$

This integral is identical to (17) except for the different normal bundles N occurring in the definition of $R_{g,h}$.

The method in Section 2 to compute (17) also yields a computation of (31). As remarked after equation (26), the bundle N affects the integral (31) through $\int_C c_1(N^*)$:

$$(32) \quad C_g(h, X, \beta) = \sum_{\tau \in P(h)} \frac{(\int_C c_1(N^*))^l}{|\text{Aut}_\tau|} \prod_{i=1}^l \int_{\overline{M}_{h_i,1}} \lambda_{h_i} \lambda_{h_i-1} \left(\sum_{j=0}^{h_i-1} (-1)^j \lambda_j \psi_1^{h_i-1-j} \right).$$

Since $\int_C c_1(N^*) = 2 - 2g + K_X \cdot \beta$, Theorem 3 follows via the series analysis of Section 2.

4. Theorem 4

Let $\pi : \overline{M}_{q,1} \rightarrow \overline{M}_q$ be the universal curve (for $q \geq 2$). The class ψ_1 is the Chern class of the cotangent line bundle on $\overline{M}_{q,1}$. The kappa classes are defined by $\kappa_j = \pi_*(\psi_1^{j+1})$. Define

$$\beta_{q-2} = \pi_* \left(\sum_{j=0}^{q-1} (-1)^j \lambda_j \psi_1^{q-1-j} \right) = \sum_{j=0}^{q-2} (-1)^j \lambda_j \kappa_{q-2-j}.$$

In the notation of Section 2.3, we see:

$$Q(t) = t^2/24 + \sum_{q \geq 2} t^{2q} \int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \beta_{q-2}.$$

The results of Section 2.3 applied in case $g = 0$ prove:

$$\exp(2Q(t)) = \left(\frac{t/2}{\sin(t/2)} \right)^2.$$

After taking the logarithm, we find:

$$(33) \quad Q(t) = \log \left(\frac{t/2}{\sin(t/2)} \right).$$

The right series in (33) may be expanded as

$$\log \left(\frac{t/2}{\sin(t/2)} \right) = \sum_{q \geq 1} \frac{|B_{2q}|}{(2q)(2q)!} t^{2q}$$

by Lemma 3 of [FP]. Faber has computed

$$\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \kappa_{q-2} = \frac{1}{2^{2q-1}(2q-1)!!} \frac{|B_{2q}|}{2q}$$

from Witten's conjectures/ Kontsevich's theorem [F2]. It is known $\mathcal{R}^{q-2}(M_q)$ is exactly 1 dimensional ([F2], [L]). Since $\lambda_q \lambda_{q-1}$ vanishes when restricted to $\partial \overline{M}_q$, we find

$$\beta_{q-2} = \frac{\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \beta_{q-2}}{\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \kappa_{q-2}} \cdot \kappa_{q-2}.$$

Theorem 4 now follows from the computation:

$$\frac{\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \beta_{q-2}}{\int_{\overline{M}_q} \lambda_q \lambda_{q-1} \cap \kappa_{q-2}} = \frac{2^{q-1}}{q!}.$$

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